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WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR MULTIDIMENSIONAL DIFFERENCE EQUATIONS IN RATIONAL CONES

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Abstract: The Cauchy problem is studied for a multidimensional difference equation in a class of functions defined at the integer points of a rational cone. We give an easy-to-check condition on the coefficients of the characteristic polynomial of the equation sufficient for solvability of the problem. A multidimensional analog of the condition ensuring stability of the Cauchy problem is stated on using the notion of amoeba of an algebraic hypersurface.

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§ 1. Introduction

Let us exhibit the general statement of the problem. Given complex-valued functions $f(x) = f(x_1, \dots, x_n)$ of integer variables $x_1, \dots, x_n$, define the translation δ_j in the variables x_j,

$$\delta_j f(x) = f(x_1, \dots, x_{j-1}, x_j + 1, x_{j+1}, \dots, x_n),$$

and a polynomial difference operator of the form

$$P(\delta) = \sum_{\omega \in \Omega} c_\omega \delta^\omega,$$

where $\Omega \subset \mathbb{Z}^n$ is a finite subset of the n-dimensional integer lattice $\mathbb{Z}^n$, $\delta^\omega = \delta_1^{\omega_1} \cdot \ldots \cdot \delta_n^{\omega_n}$ and c_ω are constant coefficients of the difference operator. It is natural to assume that the functions $f(x)$ in the domain of $P(\delta)$ are defined on a set K containing all translations of $\Omega \subset K$ as well as those by the elements $x \in K$. We take a rational cone as this set.

Let $a^1, \dots, a^n$ be linearly independent vectors with integer coordinates $a^j = \big(a_1^j, \dots, a_n^j\big)$, $a_i^j \in \mathbb{Z}$. A *rational cone* (see [1, 2]) *generated by* $a^1, \dots, a^n$, we call the set

$$K = \{x \in \mathbb{R}^n : x = \lambda_1 a^1 + \dots + \lambda_n a^n,\ \lambda_j \in \mathbb{R}_+,\ j = 1, \dots, n\}.$$

Note that this cone is *simplicial*; i.e., its every element expands in the generators uniquely.

Define the partial order $\underset{K}{\geq}$ between the points $u, v \in \mathbb{R}^n$ as follows:

$$u \underset{K}{\geq} v \Leftrightarrow u \in v + K,$$

where $v + K$ is the translation of K by v. Moreover, we write $u \underset{K}{\ngeq} v$ whenever $u - v \notin K$.

Given $m \in K \cap \mathbb{Z}^n$, put $X_m = \{x \in K \cap \mathbb{Z}^n : x \underset{K}{\ngeq} m\}$. We call the points of X_m *initial* (*boundary*). We consider the difference equations of the form

$$P(\delta) f(x) = g(x), \quad x \in K \cap \mathbb{Z}^n, \tag{1}$$

where $f(x)$ is unknown, while $g(x)$ and $\varphi(x)$ are given functions on $K \cap \mathbb{Z}^n$. We can state the problem:

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Find a function $f(x)$ satisfying (1) *and agreeing with a given function $\varphi(x)$ on X_m*, i.e.,

$$f(x) = \varphi(x), \quad x \in X_m. \tag{2}$$

It is natural to call this problem the *Cauchy problem* for (1) and condition (2) serves in this case as the *initial data* of the Cauchy problem. The Cauchy problem is called *solvable* if it has a unique solution. For a given cone K and a set of translations Ω, solvability of the problem depends on $m \in \Omega$ and the coefficients c_ω of $P(\delta)$. In the one-dimensional case, it is known for $K = \mathbb{R}_+$ (see, for instance, [3]) that the Cauchy problem (1), (2) is solvable if and only if $P(\delta) = \sum_{\omega=0}^{m} c_\omega \delta^\omega$; i.e., the point m is an order of the difference operator (the degree of the characteristic polynomial). Thus, for $n > 1$, the condition $m \in \Omega$ seems to be natural. However, this condition is not sufficient (see [4]) and the question on a relevant setting of the Cauchy problem ensuring existence and uniqueness of solutions is not trivial. The conventional situation for the combinatorial analysis problems is the case when $K \cap \mathbb{Z}^n = \mathbb{Z}^n_+$. This case was treated in [5], where sufficient conditions of solvability of the Cauchy problem (see [5, Theorem 5, p. 55]) are presented. In our notations, the question of conditions on m and Ω ensuring solvability of the Cauchy problem is considered in [5]. In Section 1 this question is examined in more general situation; namely, in the case when solutions to (1), (2) are sought in the intersection of the rational cone K containing the set Ω of translations with the integer lattice $\mathbb{Z}^n$. In this case we need to define the notion of the order of $P(\delta)$. The cone $K^* = \{k \in \mathbb{R}^n : \langle k, x\rangle \geq 0,\ x \in K\}$, where $\langle k, x\rangle = k_1x_1 + \dots + k_nx_n$, is called *dual to* K. Denote the set of its interior points by $\overset{\circ}{K}{}^*$ and fix $\nu \in \overset{\circ}{K}{}^* \cap \mathbb{Z}^n$. Given $x \in K \cap \mathbb{Z}^n$, the nonnegative number $|x|_\nu = \langle \nu, x\rangle$, is referred to as the *weighted homogeneous degree of* z^x. The *weighted homogeneous degree of the Laurent polynomial* $Q(z) = \sum_x q_x z^x$ is defined by the formula $\deg_\nu Q(z) = \max_x |x|_\nu$.

The Laurent polynomial $P(z) = \sum_{\omega\in\Omega} c_\omega z^\omega$ is called the *characteristic polynomial* of (1).

By the *order* d_ν *of* $P(\delta)$, we mean the weighted homogeneous degree $\deg_\nu P(z)$ of the characteristic polynomial, i.e., $d_\nu = \max_{\omega\in\Omega} |\omega|_\nu$. In what follows we omit the subscript ν for d.

Denote by $P_d(\delta) = \sum_{|\omega|_\nu = d} c_\omega \delta^\omega$ the principal symbol of the difference operator $P(\delta)$.

Theorem 1. *Let $m \in \Omega$ and let $|m|_\nu = d$ be the order of a difference operator. If the coefficients c_ω of the principal symbol of $P_d(\delta)$ satisfy the condition*

$$|c_m| > \sum_{|\omega|_\nu = d,\, \omega\neq m} |c_\omega|, \tag{3}$$

then (1), (2) *is solvable.*

Note that (3) of Theorem 1 is weaker than the solvability conditions of [5, Theorem 5]. Theorem 1 also validates the solvability theorems of [6, Theorem 1; 7, Theorem 1]. The case of $K = \mathbb{R}^n_+$ is considered in [5, 6] and the general case of a rational cone in [7], however, in the latter case the principal symbol of the difference operator is the summand $P_d(\delta) = c_m\delta^m$.

If the Cauchy problem is solvable then we can define the stability notion as follows: Given a function $f(x)$ with the domain $K \cap \mathbb{Z}^n$, define the norm of $f(x)$ as $\|f\| = \sup_{x\in K\cap\mathbb{Z}^n} |f(x)|$. We call the Cauchy problem (1), (2) *stable* if there exists a constant $C > 0$ such that, for given $\varphi(x)$ and $g(x)$, the corresponding solution $f(x)$ is such that $\|f\| \leq C(\|\varphi\| + \|g\|)$.

Note that stability is defined similarly in the theory of difference schemes (see, for instance, [8, 9]).

The Cauchy problem is called *well-posed* if it is solvable and stable.

In the one-dimensional case a difference operator is written as $P(\delta) = \sum_{\omega=0}^{m} c_\omega\delta^\omega$, $c_m \neq 0$, $K = \mathbb{R}_+$, and $X_m = \{0, 1, \dots, m-1\}$. Obviously, problem (1), (2) is uniquely solvable, and stability is reduced (see, for instance, [10]) to the property of the characteristic polynomial $P(z) = \sum_{\omega=0}^{m} c_\omega z^\omega$ of (1):

(∗) the interior of the unit disk $\{|z| \geq 1\}$ of the complex plane does not contain the roots of the characteristic equation $P(z) = 0$.

It follows (see [10]) from the form of the general solution to the homogeneous equation which is the sum of the elementary solutions $p_j(x)\lambda_j^x$. Here $p_j(x)$ is a polynomial in x whose degree is less than the multiplicity of the root λ_j of $P(z)$.

Asymptotics of solutions to the difference equation and different stability notions as well are studied within the framework of discrete dynamical systems. In the theory of digit signals processing stability is the most important feature of a recursive digital filter, and it is reduced to the question of convergence of the series of the modules of the Taylor coefficients of the transfer function of a filter (see [11]). For $n = 2$, the stability problem of a recursive digital filter is studied in [12].

In Section 2 we state a multidimensional analog of $(*)$ ensuring its stability. To formulate this condition, we need some notions and statements of the theory of amoebas of algebraic hypersurfaces (see [13, 14]).

We call the convex hull in $\mathbb{R}^n$ of Ω the *Newton polyhedron* N_P of a polynomial $P(z) = \sum_{\omega\in\Omega} c_\omega z^\omega$. The image of the set V of zeros of $P(z)$ under the mapping $\mathrm{Log} : z = (z_1, \dots, z_n) \to (\log|z_1|, \dots, \log|z_n|) = \mathrm{Log}\,|z|$ we call an *amoeba* $\mathscr{A}$.

The complement of an amoeba $\mathbb{R}^n\backslash\mathscr{A}$ consists of finitely many connected open components $\{E\}$. Between this collection and the points in $N_P \cap \mathbb{Z}^n$, there exists an injective mapping $\alpha : \{E\} \to N_P \cap \mathbb{Z}^n$ [14, Theorem 3.4.10] which fact allows us to "numerate" the components by integer points in $N_P \cap \mathbb{Z}^n$. If a point $\alpha \in N_P \cap \mathbb{Z}^n$ of the Newton polygon N_P corresponds to a connected component of the complement E_α, then the dual cone $C_\alpha = \{s \in \mathbb{R}^n : \max_{x\in N_P}\langle s, x\rangle = \langle s, \alpha\rangle\}$ is asymptotic for it. The latter means that this components contains the translation $u + C_\alpha \subset E_\alpha$ of the asymptotic cone together with a point $u \in E_\alpha$ itself and every cone containing C_α does not possess this property.

Observe that the cones of connected components of the complement of the amoeba corresponding to the vertices of the polygon N_P have a nonempty interior.

Theorem 2. *Let $m \in \Omega$, $|m|_\nu = d$ be the order of the difference operator and let the cone C_m dual to a point m contains the cone K^* dual to K. Then* (1), (2) *is stable if and only if the component E_m of the complement of the amoeba contains zero, i.e.,*

$(**)$ $0 \in E_m$.

For $n = 1$, condition $(**)$ coincides with $(*)$. Assume that λ_j are the roots of the characteristic equation $P(\lambda_j) = 0$ and $|\lambda_1| \le |\lambda_2| \le \dots \le |\lambda_m|$; in this case $\mathscr{A} = \{\log|\lambda_j|\}_{j=1}^m$, $\mathbb{R}\backslash\mathscr{A}$ is the union of intervals and $E_m = (\log|\lambda_m|, \infty)$. From $(**)$ it follows that $0 \in E_m$, hence, $|\lambda_m| < 1$. Thus, the component of the complement of the amoeba E_m for $n > 1$ plays a role of the interior of the unit disk.

Note that, for $K \cap \mathbb{Z}^n = \mathbb{Z}^n_+$, Theorem 2 is proven in [4, Theorem 1].

§ 2. Solvability of the Cauchy Problem

The proof of Theorem 1 is reduced to solvability of an infinite system of linear equations with infinitely many variables. It has a particular form; namely, every equation contains finitely many unknowns $f(x)$. This system is consistent whenever every subsystem of finitely many equations is consistent (see, for instance, [15, Chapter 6, Lemma 6.3.7]). Next, a sequence of finite subsystems of (1), (2) is constructed. These subsystems are such that every subsystem contains all equations of the previous subsystem. The above-mentioned lemma states that the consistency of every subsystem justifies consistency of the whole system (1), (2).

We present the algorithm of ordering the equations and "unknowns" in (1), (2).

Introduce the lexicographical order relation $\underset{K}{\prec}$ between integer points of the rational cone K. Each $x \in K \cap \mathbb{Z}^n$ is a linear combination $x = \lambda_1 a^1 + \dots + \lambda_n a^n$, with $a^1, \dots, a^n$ the generators of K. Since the cone K is simplicial, this representation is unique. Denote by $\pi^j x = \lambda_j a^j$ the projection of x onto a^j. Given $x, y \in K \cap \mathbb{Z}^n$, define the relation $\underset{K}{\prec}$ of lexicographical order in a rational cone as follows: By definition, if $\pi^j x = \pi^j y$ for some $i \in \{1, 2, \dots, n\}$ and all $j < i$ and $\pi^i x \underset{K}{<} \pi^i y$ then $x \underset{K}{\prec} y$.

Fix a vector $\nu \in \overset{\circ}{K}{}^* \cap \mathbb{Z}^n$ and consider the linear function $\langle \nu, x\rangle$, $x \in K$ (in x). Its range on $K \cap \mathbb{Z}^n$ can be ordered: $s_1 < s_2 < \dots < s_n < \cdots$ (it is possible since this simplicial cone K is salient; i.e., K does not include straight lines). Denote this range by S_ν. Note that $S_\nu \subset \mathbb{Z}_+$, since $\nu \in \overset{\circ}{K}{}^*$. A weighted lexicographical order $\lhd$ on the set of integer points of K is defined as follows: Given $x, y \in K \cap \mathbb{Z}^n$, the relation $x \lhd y$ means that $\langle \nu, x\rangle < \langle \nu, y\rangle$ and if $\langle \nu, x\rangle = \langle \nu, y\rangle$ then $x \underset{K}{\prec} y$.

Take an arbitrary $s_j \in S_\nu$. The unknowns are enumerated by the elements of $J_j = \{y \in K \cap \mathbb{Z}^n : \langle \nu, y\rangle \leq s_j\}$. Divide J_j into the two sets

$$J_j \cap X_m = \{y \in K \cap \mathbb{Z}^n : m \underset{K}{\not\leq} y,\ \langle \nu, y\rangle \leq s_j\},$$

$$J_j \backslash (J_j \cap X_m) = \{y \in K \cap \mathbb{Z}^n : m \underset{K}{\leq} y,\ \langle \nu, y\rangle \leq s_j\}.$$

Note that $J_j \backslash (J_j \cap X_m) = m + I_j$, where $I_j = \{x \in K \cap \mathbb{Z}^n : \langle \nu, x\rangle \leq s_j - \langle \nu, m\rangle\}$. Equations (1) are enumerated by elements of I_j and equations (2) by elements of $I_{m,j} = \{\mu \in X_m : \langle \nu, \mu\rangle \leq s_j\}$; in this case we assign the "numbers" of points $m + x \in J_j$ to the points x of I_j. If $\#M$ stands for the number of elements of a finite set M then $\#I_j + \#I_{m,j} = \#J_j$. Thus, we obtain the system of linear equations with the unknowns $f(y)$, $y \in J_j$:

$$\sum_{\omega \in \Omega} c_\omega f(x + \omega) = g(x), \quad x \in I_j, \tag{4}$$

$$f(\mu) = \varphi(\mu), \quad \mu \in I_{m,j}. \tag{5}$$

Denote by $\Delta_{m,j}$ the determinant of (4), (5).

Example 1. Consider the difference equation (1), with the cone K generated by the vectors $(1, 1)$ and $(-1, 1)$, i.e.,

$$c_{0,0} f(x_1, x_2) + c_{-1,1} f(x_1 - 1, x_2 + 1) + c_{0,1} f(x_1, x_2 + 1)$$
$$+ c_{1,1} f(x_1 + 1, x_2 + 1) = 0, \quad x \in K \cap \mathbb{Z}^2.$$

In this case $\Omega = \{(0, 0), (-1, 1), (0, 1), (1, 1)\}$. We can state the Cauchy problem for $m = (0, 1)$ as follows: we look for a solution satisfying the initial data

$$f(x_1, x_2) = \varphi(x_1, x_2), \quad (x_1, x_2) \in X_m,$$

where $X_m = \{(\mu_1, \mu_2) \in K \cap \mathbb{Z}^2 : (\mu_1, \mu_2) \underset{K}{\not\geq} (0, 1)\}$.

Expose the form of the system (4), (5) for the above Cauchy problem with $\nu = (0, 1)$ and $s_2 = 2$. We have

$$c_{0,0} f(x_1, x_2) + c_{-1,1} f(x_1 - 1, x_2 + 1) + c_{0,1} f(x_1, x_2 + 1) + c_{1,1} f(x_1 + 1, x_2 + 1) = 0, x \in I_2, \tag{6}$$

$$f(\mu_1, \mu_2) = \varphi(\mu_1, \mu_2), \mu \in I_{(0,1),2}, \tag{7}$$

where the set $J_2 = \{(0, 0),\ (-1, 1),\ (0, 1),\ (1, 1),\ (-2, 2),\ (-1, 2),\ (0, 2),\ (1, 2),\ (2, 2)\}$ enumerates the unknowns $f(y_1, y_2)$ and $I_{(0,1),2} = \{(0, 0), (-1, 1), (1, 1), (-2, 2), (2, 2)\}$, $I_2 = \{(\bar{0}, \bar{0}), (-\bar{1}, \bar{1}), (\bar{0}, \bar{1}), (\bar{1}, \bar{1})\}$. The bar over the coordinates of $(\bar{x}_1, \bar{x}_2)$ means that this point has the same number as $(x_1, 1 + x_2)$. Write the set $I_2 + I_{(0,1),2}$ in order of its elements, i.e., $I_2 + I_{(0,1),2} = \{(0, 0), (-1, 1), (\bar{0}, \bar{0}), (1, 1), (-2, 2), (\bar{-1}, \bar{1}), (\bar{0}, \bar{1}), (\bar{1}, \bar{1}), (2, 2)\}$. The determinant of the matrix of (6), (7) takes the form

$$\Delta_{(0,1),2} = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c_{0,0} & c_{-1,1} & c_{0,1} & c_{1,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & c_{0,0} & 0 & 0 & c_{-1,1} & c_{0,1} & c_{1,1} & 0 & 0 \\ 0 & 0 & c_{0,0} & 0 & 0 & c_{-1,1} & c_{0,1} & c_{1,1} & 0 \\ 0 & 0 & 0 & c_{0,0} & 0 & 0 & c_{-1,1} & c_{0,1} & c_{1,1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}.$$

Lemma 1. *Problem* (1), (2) *is solvable for all* $\varphi(x)$ *and* $g(x)$ *if and only if the determinants* $\Delta_{m,j}$ *do not vanish for all* $j \in \mathbb{Z}_+$.

PROOF. Demonstration is similar to that of Theorem 2 in [6], only we should use S_ν rather than $\mathbb{Z}_+$.

The coefficients c_ω of the characteristic polynomial $P(z)$ enter into the determinants $\Delta_{m,j}$, $j \in \mathbb{Z}_+$, responsible for solvability of the Cauchy problem. Demonstrate that solvability of the Cauchy problem (4), (5) depends really only on the coefficients of the principal symbol of $P_d(\delta)$.

Denote

$$J'_q = \{y \in K \cap \mathbb{Z}^n : \langle \nu, y\rangle = s_q\}, \quad I'_q = \{x \in K \cap \mathbb{Z}^n : \langle \nu, x\rangle = s_q - \langle \nu, m\rangle\},$$

$$I'_{m,q} = \{\mu \in X_m : \langle \nu, \mu\rangle = s_q\}.$$

As is easily seen, $\#I'_q + \#I'_{m,q} = \#J'_q = N'_q$, where N'_q is the number of nonnegative integer solutions to the equation $\langle \nu, y\rangle = s_q$. Note that $\sum_{i=1}^{j} N'_{s_i} = N_{s_j}$ is the number of nonnegative integer solutions to the inequality $\langle \nu, y\rangle \le s_j$.

Given $s_q \in S_\nu$, $q \le j$, denote by $D_{m,q}$ the minors of the determinant $\Delta_{m,q}$ composed of its rows corresponding to the equations

$$\sum_{\omega\in\Omega} c_\omega f(x+\omega) = g(x), \quad x \in I'_q,$$

$$f(\mu) = \varphi(\mu), \quad \mu \in I'_{m,q}.$$

and columns corresponding to the unknowns $f(y)$, with $y \in J'_q$.

Note that the determinants $D_{m,q}$ contain only the coefficients of the principal symbol of $P_d(\delta)$.

EXAMPLE 2. For the difference operator $P(\delta_1, \delta_2)$ of Example 1 and $q = 0, 1, 2$, we have

$$D_{(0,1),0} = 1, D_{(0,1),1} = \begin{vmatrix} 1 & 0 & 0 \\ c_{-1,1} & c_{0,1} & c_{1,1} \\ 0 & 0 & 1 \end{vmatrix},$$

$$D_{(0,1),2} = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ c_{-1,1} & c_{0,1} & c_{1,1} & 0 & 0 \\ 0 & c_{-1,1} & c_{0,1} & c_{1,1} & 0 \\ 0 & 0 & c_{-1,1} & c_{0,1} & c_{1,1} \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}.$$

The connections between $\Delta_{m,s}$ and $D_{m,q}$ are described in

Lemma 2. *If* $j \in \mathbb{Z}_+$, *then*

$$\Delta_{m,j} = \prod_{s_q \in S_\nu,\, q \le j} D_{m,q}.$$

The proof is given in [6, Lemma 1].

Lemmas 1 and 2 imply immediately that

Lemma 3. *Problem* (4), (5) *is solvable for all* $\varphi(x)$ *and* $g(x)$ *if and only if* $D_{m,j} \neq 0$ *for all* $j \in \mathbb{Z}_+$.

PROOF OF THEOREM 1. In accord with Lemma 2, we have

$$\Delta_{m,j} = \prod_{s_q \in S_\nu,\, q \le j} D_{m,q},$$

where $D_{m,q}$ are the main minors of the determinant $\Delta_{m,j}$ of order N'_q. The determinants $D_{m,q}$ depend only on the coefficients c_ω of the principal symbol of $P_d(\delta)$. The principal diagonal of $\Delta_{m,j}$ contains elements equal to unity and the coefficient c_m. Indeed, for ordering defined at the beginning of Section 2 for the unknowns $f(y)$ and equations (4), (5), the "number" of an equation in (5) is coded by $\mu \in I_{m,j}$.

At the row of the determinant $\Delta_{m,j}$ with the same number as μ, the only nonzero (equal to unity) element is the coefficient before the unknown function $f(\mu)$, this unknown function has the same number as $\mu \in J_j$. Hence, this nonzero element lies at the diagonal of the determinant.

The element c_m belongs to the row of $\Delta_{m,j}$ with the same number as $x \in I_j$. It can be explained by the fact that in accord with the method of ordering the point x and the unknown function $f(x+m)$ have the same numbers, the coefficient of this function in (4) is equal to c_m.

If (3) holds then $D_{m,q}$ is the determinant of a matrix with diagonal dominance (we say that a square matrix $A = \|a_{ij}\|$ *possesses the diagonal dominance* if $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$, $i = 1, 2, \ldots, n$), hence, $D_{m,q} \neq 0$ (see, for instance, [16]). By Lemma 3, the problem (4), (5) is solvable for every j which justifies solvability of (1), (2). $\square$

§ 3. Stability of the Cauchy Problem

In this section we examine stability of the Cauchy problem (1), (2). It is closely connected with the properties of the fundamental solution since every solution is expressed through this solution and the input data (the initial condition and the right-hand side (see Lemma 4). Moreover, stability is equivalent to absolute summability of the fundamental solution (Lemma 5). Under the conditions of Theorem 2, the fundamental solution is composed of the coefficients of the Laurent series of the function $\frac{1}{P(z)}$. Lemmas 6 and 7 are required to study convergence of this series.

Note that condition (3) ensures solvability of the Cauchy problem but is insufficient for stability. For example, consider the difference equation

$$f(x+1, y+1) - 3f(x, y+1) + f(x-1, y+1) + f(x, y) = g(x, y),\ \ (x, y) \in K \cap \mathbb{Z}^2,$$

where K is the cone generated by $(1, 1)$ and $(-1, 1)$, while the initial data are equal to $f(k, k) = f(-k, k) = 1$, $k = 0, 1, 2, \ldots$. Put

$$g(x) = \begin{cases} -3, & \text{if } x = 0, \\ -1, & \text{if } x \neq 0. \end{cases}$$

If $\nu = (0, 1)$ then $d = 1$. The point $m = (0, 1)$ satisfies (3), the problem has the unique solution

$$f(x, y) = y + 1 - |x|,$$

but it is not stable since a solution is unbounded ($f(0, y) = y + 1$).

A *fundamental solution to* (1), (2) is a solution $\mathscr{P}_m(x)$ of (1) with the right-hand side of the form

$$g(x) = \delta_0(x) = \begin{cases} 0, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0, \end{cases}$$

such that $\mathscr{P}_m(x) = 0$ for $x \in X_m$.

Lemma 4. *If problem* (1), (2) *is solvable for all* $g(x)$ *and* $\varphi(x)$ *then, for* $x \in K \cap \mathbb{Z}^n$, *its solution is written as*

$$f(x) = \sum_{y \in X_m} \varphi(y) \sum_{\mu \not\leq_K y} c_\mu \mathscr{P}_m(x + \mu - y) + \sum_{y \in K \cap \mathbb{Z}^n} g(y) \mathscr{P}_m(x - y), \tag{8}$$

where $\mathscr{P}_m$ *is a fundamental solution to* (1), (2). *In this case, for every fixed* $x \in K \cap \mathbb{Z}^n$, *the number of summands on the right -hand side of* (8) *is finite.*

Proof. This is similar to that of Theorem 3 in [6], but we should take the rational cone K rather than $\mathbb{R}^n_+$.

A function $f(x)$ of the discrete argument $x \in X$ is called *absolutely summable* if the series $\sum_{x\in X} |f(x)|$ converges.

Lemma 5. *The Cauchy problem* (1), (2) *is stable if and only if its fundamental solution* $\mathscr{P}_m(x)$, $x \in K \cap \mathbb{Z}^n$ *is absolutely summable.*

Proof. Necessity. Find a solution to (1), (2) for the initial data $\varphi(x) \equiv 0$ and the right-hand side $g(x)$ constructed for an arbitrary $x_1 \in K \cap \mathbb{Z}^n$ as follows:

$$g_{x_1}(x) = \begin{cases} \frac{\overline{\mathscr{P}_m(x_1-x)}}{|\mathscr{P}_m(x_1-x)|}, & \text{if } \mathscr{P}_m(x_1 - x) \neq 0, \\ 0, & \text{if } \mathscr{P}_m(x_1 - x) = 0. \end{cases}$$

By Lemma 4, a solution $f_{x_1}(x)$ to (1), (2) with these data is representable as

$$f_{x_1}(x) = \sum_{y \in K \cap \mathbb{Z}^n} g_{x_1}(y) \mathscr{P}_m(x - y).$$

For $x = x_1$, we infer

$$f_{x_1}(x_1) = \sum_{y \in K \cap \mathbb{Z}^n} \frac{\overline{\mathscr{P}_m(x_1 - y)}}{|\mathscr{P}_m(x_1 - y)|} \mathscr{P}_m(x_1 - y) = \sum_{y \in K \cap \mathbb{Z}^n} |\mathscr{P}_m(x_1 - y)| = \sum_{0 \underset{K}{\leq} y' \underset{K}{\leq} x_1} |\mathscr{P}_m(y')|.$$

Since $\|\varphi\| = 0$ and $\|g_{x_1}\| \leq 1$, stability of (1), (2) yields

$$|f_{x_1}(x_1)| = \sum_{0 \leq y' \leq x_1} |\mathscr{P}_m(y')| \leq C$$

for some $C > 0$ and arbitrary $x_1 \in K \cap \mathbb{Z}^n$.

Sufficiency. By Lemma 4, $f(x) = f_0(x) + f^*(x)$, where

$$f_0(x) = \sum_{y \in X_m} \varphi(y) \sum_{\mu \underset{K}{\not\leq} y} c_\mu \mathscr{P}_m(x + \mu - y)$$

is a solution to the homogeneous problem (1), (2) and

$$f^*(x) = \sum_{y \in K \cap \mathbb{Z}^n} g(y) \mathscr{P}_m(x - y)$$

is a solution to the inhomogeneous Cauchy problem with the zero initial data.

Put $C_1 = \max_{\omega \in \Omega} |c_\omega|$ and estimate $|f_0(x)|$ and $|f^*(x)|$ as follows:

$$|f_0(x)| \leq \sum_{y \in X_m} |\varphi(y)| \sum_{\mu \underset{K}{\not\leq} y} |c_\mu| |\mathscr{P}_m(x + \mu - y)|$$

$$\leq C_1 \|\varphi\| \sum_{y \in X_m} \sum_{\mu \underset{K}{\not\leq} y} |\mathscr{P}_m(x + \mu - y)| \leq C_1 \|\varphi\| \sum_{0 \underset{K}{\leq} y \underset{K}{\leq} x+m} |\mathscr{P}_m(y)|.$$

Next, we have

$$|f^*(x)| \leq \sum_{y \in K \cap \mathbb{Z}^n} |g(y)| |\mathscr{P}_m(x - y)| \leq \|g\| \sum_{0 \underset{K}{\leq} y \underset{K}{\leq} x} |\mathscr{P}_m(y)|.$$

Since the fundamental solution is absolutely summable, i.e.

$$\sum_{x \in K \cap \mathbb{Z}^n} |\mathscr{P}_m(x)| \leq C_2$$

for some $C_2 > 0$; we obtain $\|f\| \leq C_1 \|\varphi\| C_2 + \|g\| C_2 \leq C(\|\varphi\| + \|g\|)$. $\square$

To prove Theorem 2, we need two additional statements concerning with convergence of the Laurent series supported in rational cones.

Let $q = (q_1, \dots, q_n) \in \mathbb{C}^n$. Consider the sum of a geometric progression supported in a rational cone K. Let

$$\Lambda = \{x \in \mathbb{Z}^n : x = \lambda_1 a^1 + \dots + \lambda_n a^n,\ \lambda_i \in \mathbb{Z},\ i = 1, \dots, n\}$$

be a sublattice of the lattice $\mathbb{Z}^n$ generated by $a^1, \dots, a^n$. Assume that $\tau = a^1 + \dots + a^n$, $\Pi_\tau = \{x \in \mathbb{R}^n : 0 \underset{K}{\leq} x \underset{K}{<} \tau\}$ is a half-open parallelotope. Denote by $\{v\}$ the set of points v with integer coordinates lying in Π_τ. Obviously,

$$\bigcup_{v \in \Pi_\tau \cap \mathbb{Z}^n} (v + \Lambda) = \mathbb{Z}^n.$$

Lemma 6. *The geometric progression $\sum_{x \in K \cap \mathbb{Z}^n} q^x$ converges for all q such that $|q^{a^j}| < 1$, $j = 1, \dots, n$, and its sum is equal to*

$$\sum_{x \in K \cap \mathbb{Z}^n} q^x = \Big(\sum_{v \in \Pi_\tau \cap \mathbb{Z}^n} q^v \Big) \prod_{i=1}^{n} (1 - q^{a^i})^{-1}.$$

Proof. Indeed, since $\bigcup_{v \in \Pi_\tau \cap \mathbb{Z}^n} (v + \Lambda) = \mathbb{Z}^n$ and $x = \lambda_1 a^1 + \dots + \lambda_n a^n$, we have

$$\sum_{x \in K \cap \mathbb{Z}^n} q^x = \sum_{v \in \Pi_\tau \cap \mathbb{Z}^n} \sum_{x \in K \cap \Lambda} q^{v+x} = \sum_{v \in \Pi_\tau \cap \mathbb{Z}^n} q^v \sum_{\lambda \in \mathbb{Z}^n_+} q^{\lambda_1 a^1 + \dots + \lambda_n a^n}.$$

For $|q^{a^j}| < 1$, $j = 1, \dots, n$, the series $\sum_{\lambda \in \mathbb{Z}^n_+} (q^{a^1})^{\lambda_1} \dots (q^{a^n})^{\lambda_n}$ converges and thereby

$$\sum_{x \in K \cap \mathbb{Z}^n} q^x = \Big(\sum_{v \in \Pi_\tau \cap \mathbb{Z}^n} q^v \Big) \prod_{i=1}^{n} (1 - q^{a^i})^{-1}. \quad \square$$

We can formulate a version of the Abel lemma (see [17]) for the Laurent series whose supports lie in a rational cone. Denote the interior of the dual cone by $\operatorname{Int} K^*$.

Lemma 7. *If all members of the Laurent series K*

$$\sum_{x \in K \cap \mathbb{Z}^n} \frac{f(x)}{z^x} \tag{9}$$

supported in a rational cone are bounded at z_0, then the series is convergent for z such that $z \in \operatorname{Int}(z_0 + \operatorname{Log}^{-1} K^*)$.

Proof. Assume that $\{\frac{f(x)}{z_0^x}\}_{x \in K \cap \mathbb{Z}^n}$ is bounded by $M > 0$ and $x = \lambda_1 a^1 + \dots + \lambda_n a^n$. In this case

$$\left| \frac{f(x)}{z^x} \right| = \left| \frac{f(x)}{z_0^x} \right| \left| \frac{z_0^x}{z^x} \right| \leq M \left| \frac{z_0^{\lambda_1 a^1 + \dots + \lambda_n a^n}}{z^{\lambda_1 a^1 + \dots + \lambda_n a^n}} \right| \leq M |(q^{a^1})^{\lambda_1} \dots (q^{a^n})^{\lambda_n}|,$$

where $q = \left(\frac{z_0}{z}, \dots, \frac{z_0}{z} \right)$. By Lemma 6, for $|q^{a^j}| < 1$, $j = 1, \dots, n$, series (9) converges absolutely for z such that $|z_0^{a^j}| < |z^{a^j}|$, $j = 1, \dots, n$. Taking the logarithm we see that the series converges for z satisfying $\langle \operatorname{Log} z, a^j \rangle > \langle \operatorname{Log} z_0, a^j \rangle$, $j = 1, \dots, n$. The latter means that $\operatorname{Log} z \in \operatorname{Int}(\operatorname{Log} z_0 + K^*)$ or $z \in \operatorname{Int}(z_0 + \operatorname{Log}^{-1} K^*)$. $\square$

Proof of Theorem 2. Note that the condition of Theorem 2 for $m \in \Omega$ means that m is a vertex of the Newton polyhedron N_P. Indeed, assume on the contrary that m is not a vertex of the Newton polyhedron. Since $m \in \Omega$, either m belong to the interior of the polyhedron or its zero face Γ. Hence, the dimension of C_m is equal to $n - \dim \Gamma$ (see [13, p. 46]), i.e., it is less than n (in the first case C_m is equal to $\{0\}$). Therefore, C_m cannot include the cone K^*, since it is n-dimensional.

In particular, since m is a vertex of the Newton polyhedron, the corresponding component of the complement E_m of the amoeba is not empty. Moreover, since $|m|_\nu = d$, the principal symbol of the difference operator consists of one summand $P_d(\delta) = c_m \delta^m$. Thereby, the conditions of Theorem 1 are fulfilled, i.e., there exists a unique solution to the Cauchy problem. Note also that the conditions of Theorem 2 yield $N_P \subset \Pi_m = \{t : 0 \underset{K}{\leq} t \underset{K}{\leq} m\}$.

Demonstrate that the condition $K^* \subset C_m$ for a point m of the Newton polyhedron N_P ensures not only existence and uniqueness of the fundamental solution to the Cauchy problem (by Theorem 1), but also the possibility of its constructive representation. Indeed, the characteristic polynomial is written as follows: $P(z) = \sum_{0 \underset{K}{\leq} \omega \underset{K}{\leq} m} c_\omega \delta^\omega$, and we can expand the rational function $\frac{1}{P(z)}$ in the Laurent series

$$\frac{1}{P(z)} = \frac{1}{c_m z^m + \sum\limits_{\omega \neq m} c_\omega z^\omega} = \frac{1}{c_m z^m (1 - \sum\limits_{\omega \neq m} \frac{\tilde{c}_\omega}{z^{m-\omega}})}.$$

Since $\omega \leq_K m$, we have $m - \omega \underset{K}{\geq} 0$, i.e., $m - \omega \in K \cap \mathbb{Z}^n$; in this case

$$\frac{1}{P(z)} = \frac{1}{c_m z^m} \sum_{k=0}^{\infty} \Big(\sum_{\omega \neq m} \frac{\tilde{c}_\omega}{z^{m-\omega}} \Big)^k = \sum_{x \in m + K_m} \frac{\widetilde{\mathscr{P}}_m(x)}{z^x}.$$

Note that $\widetilde{\mathscr{P}}_m(x)|_{X_m} = 0$ and $\operatorname{supp} \widetilde{\mathscr{P}}_m(x) \subset m + K_m$, where K_m is a cone constructed on the vectors $m - \omega$, $\omega \in \Omega$, $K_m \subset K$. The series obtained converges in a domain $\mathscr{E} \subset \mathbb{C}^n$ such that $\operatorname{Log} \mathscr{E} = E_m$ (see [13]). Accounting for the membership $K_m \subset K$, we can write the expansion as follows:

$$\frac{1}{P(z)} = \sum_{x \underset{K}{\geq} 0} \frac{\widetilde{\mathscr{P}}_m(x)}{z^x}.$$

Show that the coefficients $\widetilde{\mathscr{P}}_m(x)$ of the expansion of $\frac{1}{P(z)}$ are a fundamental solution to (1), (2). Multiplying the last inequality by $P(z)$ and taking it into account that $\widetilde{\mathscr{P}}_m(x)|_{X_m} = 0$, we derive the identity

$$1 = \Big(\sum_{0 \underset{K}{\leq} \omega \underset{K}{\leq} m} c_\omega z^\omega \Big) \Big(\sum_{x \underset{K}{\not\geq} \omega} \frac{\widetilde{\mathscr{P}}_m(x)}{z^x} + \sum_{x \underset{K}{\geq} \omega} \frac{\widetilde{\mathscr{P}}_m(x)}{z^x} \Big) = \sum_{0 \underset{K}{\leq} \omega \underset{K}{\leq} m} c_\omega \sum_{x \underset{K}{\geq} \omega} \frac{\widetilde{\mathscr{P}}_m(x)}{z^{x-\omega}}$$

$$= \sum_{0 \underset{K}{\leq} \omega \underset{K}{\leq} m} c_\omega \sum_{x \underset{K}{\geq} 0} \frac{\widetilde{\mathscr{P}}_m(x+\omega)}{z^x} = \sum_{x \underset{K}{\geq} 0} \sum_{0 \underset{K}{\leq} \omega \underset{K}{\leq} m} \frac{c_\omega \widetilde{\mathscr{P}}_m(x+\omega)}{z^x}.$$

Equating the coefficients of the same powers z^x, $x \in K \cap \mathbb{Z}^n$, we find that $\sum_\omega c_\omega \widetilde{\mathscr{P}}_m(x) = \delta_0(x)$ and in view of uniqueness of solutions to (1), (2) the coefficients of the Laurent expansion of the function $\frac{1}{P(z)}$ coincide with its fundamental solution $\mathscr{P}_m(x)$.

Necessity. By the condition of the theorem the Cauchy problem is stable and, hence, by Lemma 5, the numerical series $\sum_{x \in K \cap \mathbb{Z}^n} |\mathscr{P}_m(x)|$ converges. The above equality $\widetilde{\mathscr{P}}_m(x) = \mathscr{P}_m(x)$ implies that the Laurent series $\sum_{x \in K \cap \mathbb{Z}^n} \frac{\mathscr{P}_m(x)}{\xi^x}$ converges for all ξ such that $|\xi_j| = 1$, $j = 1, \ldots, n$, to $\frac{1}{P(\xi)}$, i.e., $P(\xi) \neq 0$. By Lemma 7 this series converges in the domain $\operatorname{Int}(\xi + \operatorname{Log}^{-1} K^*)$ as well, since $K^* \subset C_m$ by condition. Hence, $\operatorname{Int}(\operatorname{Log} \xi + K^*) \subset \operatorname{Int}(\operatorname{Log} \xi + C_m) \subset E_m$ and $0 = \operatorname{Log} \xi \in E_m$.

Sufficiency. Conversely, if $0 \in E_m$ then there exists $\xi \in \mathscr{E} = \operatorname{Log}^{-1} E_m$ such that $|\xi_j| = 1$, $j = 1, \ldots, n$, and the series $\sum_{x \in K \cap \mathbb{Z}^n} \frac{\mathscr{P}_m(x)}{\xi^x}$ converges absolutely; i.e. the fundamental solution is absolutely summable. By Lemma 5 (1), (2) is stable.

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